Complete Smooth Wall Velocity Profile in a Turbulent Boundary Layer

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Nomenclature

A = log-law intercept
a = parameter in Eq. (5)
b = parameter in Eq. (5)
c_f = skin-friction coefficient
$g(\Pi, \frac{y}{\delta})$ = function representing law of the wake
H = shape factor
k = log-law slope
ME = maximum error = |Eqs. (3)/(5)/(6) or (7) − Expt|/Expt
R_θ = Reynolds number based on momentum thickness
SD = relative percentage of standard deviation

$$\sqrt{\frac{[(\text{Eqs. (3)/(5)/(6) or (7)} - \text{Expt})/\text{Expt}]^2}{N}}$$

U_∞ = freestream velocity
u = mean velocity in flow direction
u^+ = u/u_τ
u_τ = shear velocity
y = distance from the wall
y^+ = $y\, u_\tau/\nu$
ν = kinematic viscosity of the fluid
δ = boundary-layer thickness
θ = momentum thickness
γ_s = weighting function in Eq. (7)
Π = Coles wake parameter

Introduction

To predict turbulent boundary layers, it is essential to specify the initial profile in terms of the dependent variables. A good representation of the initial mean velocity profile should have some or preferably all of the following characteristics:[1] 1) should be of closed form, 2) should be explicitly expressed as a function of the distance from wall, 3) should be valid over the complete thickness of the boundary layer, and 4) should be relatively easy to evaluate.

Turbulent boundary-layer representation is usually based on the concept of inner and outer regions. Spalding[2] has shown that the experimental velocity distribution in the inner region consisting of viscous sublayer, the transition region, and the turbulent core can be represented by the relation

$$y^+ = u^+ + e^{-A}\{e^{ku^+} - 1 - [(ku^+)^2/2!] - [(ku^+)^3/3!] - [(ku^+)^4/4!]\} \cdots \quad (1)$$

Coles[3] proposed that the profile outside the sublayer could be described by a summation of the log law and a function $g(\Pi, y/\delta)$, which would represent the outer layer deviation from the law of the wall:

$$u^+ = 1/k\,(\ln y^+ + A) + g(\Pi, y/\delta) \cdots \quad (2)$$

Representation of Complete Profile

Various investigations have used Eqs. (1) and (2) in various ways to obtain the complete representation of the turbulent boundary layer.

Musker[4] proposed the following explicit closed-form expression:

$$u^+ = \underbrace{\log_{10}\left\{\frac{(y^+ + 10.6)^{9.6}}{[(y^+)^2 - 8.15\,y^+ + 86]^2}\right\}}_{\text{I}} + \underbrace{5.424\tan^{-1}\left(\frac{2y^+ - 8.15}{16.7}\right)}_{\text{II}} - \underbrace{3.52}_{\text{III}} + 2.44\left\{\Pi\left[6\frac{(y)^2}{\delta^2} - 4\frac{(y)^3}{\delta^3}\right] + \left[\frac{(y)^2}{\delta^2}\left(1 - \frac{y}{\delta}\right)\right]\right\} \cdots \quad (3)$$

Musker's representation of the inner layer was obtained by altering Spalding's function in favor of a simple expression that satisfies both continuity and momentum equations near the wall, and the outer layer deviation was represented by

$$g\left(\Pi, \frac{y}{\delta}\right) = \frac{\Pi}{k}\left[6\frac{(y)^2}{\delta^2} - 4\frac{(y)^3}{\delta^3}\right] + \frac{1}{k}\frac{(y)^2}{\delta^2}\left(1 - \frac{y}{\delta}\right) \cdots \quad (4)$$

Whitfield[5] represented the velocity profile in the following form:

$$u^+ = \frac{1}{0.09}\tan^{-1}(0.09y^+) + \left[\left(\frac{2}{c_f}\right)^{1/2} - \frac{\pi}{0.18}\right]\tanh^{1/2}\left[a\frac{(y)^b}{(\theta)^b}\right] \cdots \quad (5)$$

where a and b are parameters that are constants for a given boundary layer and are functions of c_f, H, and R_θ. The parameters a and b were determined by satisfying the requirement that velocity profiles similar to those correlated by von Doenhoff and Tetervin[6] be recovered away from the wall in the outer zone at $y/\theta = 2$ and 5, respectively.

Liakopoulos,[1] with the aim of obtaining an explicit function for the complete velocity profile, coupled the inner region with Eq. (4) and performed a series of numerical experiments to evaluate the constants for the inner region. The final expression is of the form

$$u^+ = \underbrace{\ln\left[\frac{(y^+ + 11)^{4.02}}{(y^{+2} - 7.37\,y^+ + 83.3)^{0.79}}\right]}_{\text{I}} + \underbrace{5.63\tan^{-1}[0.12\,y^+ - 0.441]}_{\text{II}} - \underbrace{3.81}_{\text{III}} + g\left(\Pi, \frac{y}{\delta}\right) \cdots \quad (6)$$

where $g(\Pi, y/\delta)$ is as given Eq. (4).

In order to obtain shear stress and eddy viscosity relations, Galbraith and Head[7] used boundary-layer equations in finite-difference form and tested their procedure against published turbulent boundary-layer development. In applying this procedure, velocity profiles were represented by Thompson's profile family

$$u/U_\infty = \gamma_s\,(u/U_\infty)_{\text{inner}} + (1 - \gamma_s) \cdots \quad (7)$$

where weighting factor γ_s is unity close to the surface and zero at the edge of the boundary layer. In Eq. (7), the velocity distribution $(u/U_\infty)_{\text{inner}}$ was represented by $u^+ = f(y^+)$ where

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the function $f(y^+)$ is defined in three regions as:

1) viscous sublayer

$$0<y^+<4, \qquad u^+=y^+ \cdots \tag{8a}$$

2) blending region

$$4<y^+<30$$

$$u^+=4.187-5.745\log_e y^+ + 5.11(\log_e y^+)^2 - 0.767(\log_e y^+)^3 \cdots \tag{8b}$$

3) the log-law region

$$y^+>30, \qquad u^+=5.50\log_{10} y^+ + 5.45 \cdots \tag{8c}$$

where γ_s distribution was prescribed as follows:

$$0<\frac{y}{\delta}\le 0.05, \qquad \gamma_s=1$$

$$0.05<\frac{y}{\delta}\le 0.3, \qquad \gamma_s=1-2.64214\left(\frac{y}{\delta}-0.05\right)^2$$

$$0.03<\frac{y}{\delta}\le 0.7, \qquad \gamma_s=4.40503\left(\frac{y}{\delta}-0.5\right)^3 - 1.8502\left(\frac{y}{\delta}-0.5\right)+0.5$$

$$0.7<\frac{y}{\delta}\le 0.95, \qquad \gamma_s=2.64214\left(0.95-\frac{y}{\delta}\right)^2$$

$$0.95<\frac{y}{\delta}\le 1, \qquad \gamma_s=0$$

Comparison with Experimental Data

In this paper, experimental results as given at the Stanford conference (1968[8]) for the following cases were used: 1) mild positive pressure gradient-Bradshaw (nos. 2501–2504), 2) moderate positive pressure gradient-Bradshaw (nos. 2601–2604), 3) severe positive pressure gradient-Stratford (nos. 5304 and 5307), and 4) negative pressure gradient-Herring and Norbury (nos. 2702 and 2706) and relaxing flow-Bradshaw (nos. 2401–2407). The experimental results were chosen so that at least five points were available in the zone for $y^+ = 100$, and in all 19 representative profiles were used. Results were analysed in the usual semilog plot for the complete profile and separately for the inner zone in the form of u^+ against y^+ for y^+ up to 100. All of the four models predicted well the experimental results for the complete profile in all of the cases except for flow with relaxation. In this particular case, far out in the outer zone, all of the four models predicted higher values when compared with experiments, as shown in Fig. 1. The results for the inner zone are consolidated in Table 1. Also indicated is the maximum error for each representation with respect to experimental values. It is seen that results obtained from Galbraith and Head's expression for the inner region generally predict the experimental values the best. For profile no. 2501, Liakopoulos' representation represents well, whereas Galbraith and Head's expression are better than Eqs. (3) and (6). For no. 5307, both Eqs. (3) and (6) are better represented when compared with Eqs. (5) and (7). For no. 2706, both Eqs. (3) and (6) predict well, whereas Eq. (7) represents better than results given by Eq. (5). It was also observed in general that the maximum deviation from experimental observation occurred at the first point measured from the wall. It is also observed that for the inner zone of all the profiles, Musker and Liakopoulos[1] representation gives almost the same results. A sample calculation for various values of y^+ is shown in Table 2. Term III in Eqs. (3) and (6) has the values of -3.52 and -3.81, respectively.

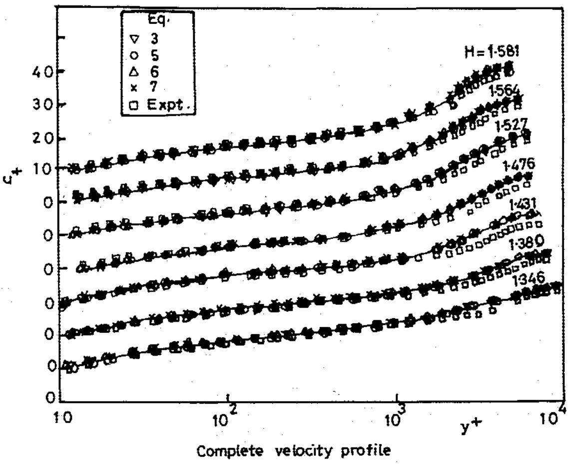

Fig. 1 Boundary layer formulas compared with data of Bradshaw (relaxing flow).

Table 1 Comparison of results for inner zone

Iden. No.		Eq. (3)	Eq. (6)	Eq. (5)	Eq. (7)
2501	SD	.061	.059	.014	.042
	ME	.085	.085	.023	.060
2502	SD	.047	.046	.036	.026
	ME	.060	.060	.066	.037
2503	SD	.049	.047	.034	.026
	ME	.070	.070	.051	.043
2504	SD	.049	.048	.032	.025
	ME	.067	.067	.049	.035
2601	SD	.123	.124	.123	.097
	ME	.261	.261	.285	.235
2602	SD	.116	.117	.114	.099
	ME	.198	.201	.220	.231
2603	SD	.138	.139	.139	.117
	ME	.356	.355	.367	.340
2604	SD	.128	.129	.130	.109
	ME	.359	.357	.370	.346
5304	SD	.161	.162	.183	.146
	ME	.341	.341	.363	.321
5307	SD	.105	.105	.125	.140
	ME	.222	.225	.247	.197
2702	SD	.024	.024	.037	.010
	ME	.038	.039	.050	.018
2706	SD	.025	.024	.060	.047
	ME	.053	.047	.089	.104
2401	SD	.147	.148	.146	.129
	ME	.410	.408	.417	.399
2402	SD	.154	.155	.152	.137
	ME	.425	.423	.430	.414
2403	SD	.171	.172	.168	.153
	ME	.448	.446	.452	.437
2404	SD	.144	.144	.135	.124
	ME	.391	.388	.391	.377
2405	SD	.131	.131	.126	.112
	ME	.352	.351	.348	.336
2406	SD	.120	.121	.137	.102
	ME	.346	.345	.345	.329
2407	SD	.115	.115	.112	.097
	ME	.327	.325	.325	.307

Table 2 y^+ vs u^+ for inner layer

	Term I		Term II		u^+	
y^+	Eq. (3)	Eq. (6)	Eq. (3)	Eq. (6)	Eq. (3)	Eq. (6)
4	7.495	7.532	−0.049	0.219	3.926	3.942
10	8.575	8.528	3.347	3.655	8.402	8.374
20	9.245	9.209	5.901	6.186	11.626	11.589
30	9.702	9.686	6.830	7.118	13.012	12.994
40	10.093	10.090	7.281	7.574	13.852	13.854
50	10.435	10.440	7.544	7.842	14.459	14.472
100	11.686	11.710	8.042	8.358	16.215	16.257

Conclusion

Galbraith and Head's expression represents the velocity profiles very well. All of the four models overestimate the velocities in the outer part of the outer zone for relaxing flows. Musker and Liakopoulos' expressions practically predict the same values in the inner zone. Whitfield's expression for the inner zone is of very simple form.

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Prediction of Radially Spreading Turbulent Jets

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Introduction

ALTHOUGH the two-equation k-ϵ turbulence model has been widely used for the prediction of various jet flows, a systematic assessment of its performance for radially spreading jets appears not to have been made. The radial free jet has been computed by Rubel[1] and the radial wall jet by Sharma and Patankar.[2] The half-width spreading rates computed by these authors are compared with those of experiment[3-6] in Table 1. Although the predictions appear satisfactory, it is important to note that while Rubel employed the standard k-ϵ model,[7] Sharma and Patankar used a different set of values for the model coefficients.

In the present work, the standard k-ϵ model is used to predict both types of radial wall jets. Some attention is also given to the prediction of plane and round jets in view of their importance as benchmarks for turbulence-model development. Calculations are made with the standard set of model coefficients, and the performance of the model is assessed by comparing the predicted spreading rates with experimental data. It will be shown in the results section that the standard model tends to underestimate the measured spreading rates of radial jets, especially for the wall-jet configuration. The discrepancies between prediction and experiment are attributed to deficiences in the ϵ-transport equation. In this study, a modification is made to this equation which results in improved predictions of radial jets.

Turbulence-Model Equations

The standard k-ϵ model determines the Reynolds stresses $-\overline{u_i u_j}$ through the isotropic eddy-viscosity hypothesis

$$-\overline{u_i u_j} = \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3} k \, \delta_{ij} \tag{1}$$

where U_i is the mean velocity component in the x_i direction, δ_{ij} is the Kronecker delta, k is the turbulent kinetic energy, and ν_t is the kinematic eddy viscosity, which is given by

$$\nu_t = C_\mu \, k^{1/2} \, L \tag{2}$$

where L is the macro length scale of the turbulent motion. The k-ϵ model determines k and L from semiempirical transport equations for k and the second turbulence property ϵ, the rate of dissipation of k. The length scale is recovered from the local values of k and ϵ by way of

$$\epsilon = C_D \, k^{3/2}/L \tag{3}$$

The turbulent kinetic energy is calculated from the following transport equation:

$$\rho \frac{\mathrm{D}k}{\mathrm{D}t} = \frac{\partial}{\partial x_i}\left(\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right) + \rho \, (P_k - \epsilon) \tag{4}$$

where μ_t is the dynamic eddy viscosity and P_k is the rate of production of k,

$$P_k = -\overline{u_i u_j} \, \frac{\partial U_i}{\partial x_j} \tag{5}$$

The dissipation rate is calculated from the following transport equation:

$$\rho \frac{\mathrm{D}\epsilon}{\mathrm{D}t} = \frac{\partial}{\partial x_i}\left(\frac{\mu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_i}\right) + \rho \, C_1 \frac{\epsilon}{k} P_k - \rho \, C_2 \frac{k^{1/2}}{L} \epsilon \tag{6}$$

The six empirical coefficients are assigned the values recom-

Table 1 Previous work: calculated and measured jet spreading rates

	Spreading rate, $d\delta_U/dx$	
Jet	Calculated (k-ϵ)	Data
Radial wall	0.080	0.085–0.095
Radial free	0.095	0.098–0.110

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